Families of variational principles for inverse and H_A hybrid problems of an S_2 stream sheet in mixed flow turbomachines

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Received November 1986 and accepted for publication December 1987

This paper deals with the semi-inverse, H_A, and inverse problems of compressible flow along an S_2 stream sheet in mixed flow turbomachines, for which families of variational principles (VPs) are derived. In the VPs for the semi-inverse problem, the distributed gas injection and/or suction along the hub walls and casing walls are accounted for. VP families for axial and radial flow turbomachines can be obtained directly from this paper as special cases. Great attention is paid to taking full advantage of natural boundary conditions and "artificial interfaces" in order to offer a perfect theoretical basis for the finite element method or other direct variational methods in computational aerodynamics of turbomachinery. This theory also provides broader and versatile ways for blading design.

Keywords: turbomachine aerodynamics; variational methods in fluids; finite element method in fluids; blading design

Introduction

The quasi-three-dimensional (quasi-3D) flow theory of turbomachinery based on Wu's S_1 (blade-to-blade) and S_2 (hub-to-tip) stream surface model[1] has found worldwide recognition and acception for practical turbomachine analysis and design. Before 1974, all numerical solutions were carried out by the streamline curvature method or by the finite difference method.[6] In the 1970s the remarkable success of the finite element method (FEM) in applied mechanics and engineering has renewed the interest of scientists and engineers in the search for variational principles (VPs) in fluid mechanics,[17] in general, and in turbomachine aerothermodynamics, in particular,[2–5,7,11,12,15] because VPs form a most reliable and useful theoretical foundation of FEM. The VP-based FEM has several advantages over the FEM based on the weighted residual procedure (including the Galerkin FEM).[18] First, the former can be directly coupled with the modern mathematical programming method. Second, functional variation with variable domain is a unique, powerful means for solving all problems with unknown boundaries, such as free surface flow, flow with cavities, shocks, free vortex sheets, and inverse and hybrid problems.[11,15]

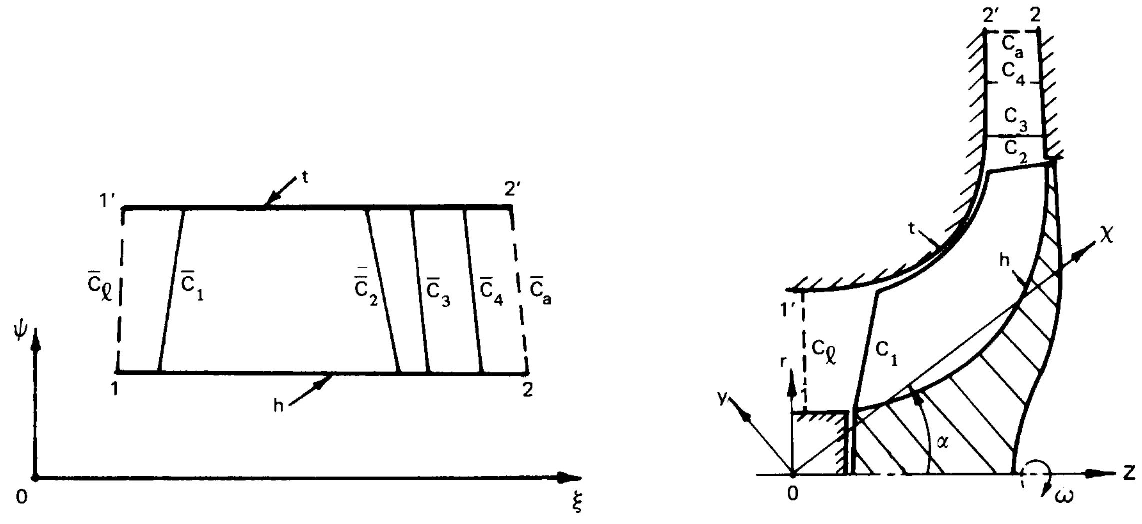
For S_2 flow the FEM was applied to the semi-inverse problem for the first time by Adler and Krimerman,[8] using an approximate VP for the Poisson equation; they claimed that an exact VP for S_2 flow could not be obtained. Subsequently, Oates *et al.*[7] developed an exact VP for the semi-inverse problem based on a crude actuator-disk model, in which the flow fields in both stators and rotors disappear, so it is of limited use for practical blading design. In Ref. 9, Hirsch and Warzee solved the semi-inverse and direct problems of S_2 flow by the Galerkin FEM, using circumferentially averaged flow equations.

At first, the hybrid problem of cascades on an arbitrary stream sheet of revolution and the corresponding VPs were suggested by Liu[5] as a unification and generalization of the traditional direct and inverse problems to meet various requirements of modern turbomachinery development Recently it has been extended to fully 3D rotor flow as well.[11,20] It is natural to handle the S_2 flow problem similarly. Thus, Liu[3] and Cai[4] developed VP families for S_2 flow in axial and radial flow turbomachines, respectively, dealing with the following three aerodynamic problems of S_2 flow:

(a) The semi-inverse problem: Given the hub and casing shapes and the distribution of $V_\varphi r$ along the S_2 stream sheet, find the flow field and the shape of the S_2 stream sheet.
(b) The hybrid problem of type "A" (abbreviated as H_A problem): Given the distribution of $V_\varphi r$ along S_2, some part of the hub and casing shapes, and the pressure distribution along the remaining part, find the shape of S_2, the hub and casing shapes, and the flow field.
(c) The complete inverse problem: This is a special case of the H_A problem, when both the hub and casing shapes are unknown but a pressure distribution along them is specified instead.

This paper is a continuation and generalization of Refs. 3 and 4 to mixed flow turbomachinery. Families of variational principles for these problems are derived herein. For this purpose, the momentum equation in an arbitrary quasi-orthogonal direction, y, is used as the principal equation.[6] In the VPs for the semi-inverse problem, the distributed gas injection and/or suction along the hub and casing walls can be accounted for for the boundary layer control, cooling, or surge-suppressing purposes. To circumvent the difficulties associated with the existence of some unknown boundary, the VPs for the hybrid problem are established on an image plane $\xi\psi$ defined by Equation 6. In contrast, in a companion paper[15] the hybrid problem is handled via functional variations with variable domain directly on the physical plane.

VP families for the same problems of axial and radial flow turbomachines presented, respectively, in Refs. 3 and 4 can be obtained directly from the results of this paper as special cases. We take full advantage of natural boundary conditions and



a) Image plane　　b) Physical plane

Figure 1 Flow field and its image on an S_2 stream sheet in mixed flow turbomachines: (a) image plane; (b) physical plane

"artificial interfaces"[16] in order to offer a sound theoretical basis for the finite element method or other direct variational methods.

Family of VPs for the semi-inverse problem

Fundamental equations of aerodynamics

For an S_2 flow in mixed flow turbomachines containing simultaneously pure axial and pure radial flow domains, a rotated rectangular (x, y) coordinate system, as shown in Figure 1, is used to avoid the indeterminacy of the soluton.[10] Then, from Refs. 1–4, we have the following aerodynamic equations:

Continuity equation:

$$\frac{\bar{\partial}\psi}{\partial x} = -Br\rho W_y, \qquad \frac{\bar{\partial}\psi}{\partial y} = Br\rho W_x \tag{1}$$

Nomenclature

A	Solution domain on S_2 stream sheet
∂A_1	Boundary segment of A with prescribed $\psi = \psi_{pr}$
∂A_2	Boundary segment of A with prescribed tangential velocity $W_t = (W_t)_{pr}$
∂A_3	Artificial interfaces[16]
B	Angular thickness of the stream sheet
C_0	p_0/ρ_0^k
C_e, C_a	Inlet and outlet, respectively
C_i	Leading/trailing edges of the blades, $i = 1, 2, \ldots$
C_v	Specific heat at constant volume
$\mathbf{F}$	Force per unit mass of fluid resulting from the pressure difference between two lateral surfaces of the stream sheet[1]
$\tilde{F}_y$	$(F_y + f_y)/W_x$
$\mathbf{f}$	Viscous force per unit mass[2,15]
k	Isentropic index
m	$(k-1)^{-1}$
p, q'	Pressure and heat input, respectively
r, φ, z	Cylindrical coordinates fixed on rotor
$\mathring{R}, R$	Rothalpy or its given distribution $\mathring{R}(\xi, \psi)$ and gas constant, respectively
S, s	Entropy and arc length, respectively
$\Delta\tilde{S}$	$(S - S_0)/C_v$
t, h	Tip and Hub annular walls, respectively
u, U	ωr, peripheral speed of blades and internal energy, respectively
$\mathbf{V}, V_\varphi$	Absolute velocity and its azimuthal component, respectively
$\mathbf{W}, W_l$	Relative velocity and its meridional component, respectively
x, y	Rectangular coordinate system on meridional plane (Figure 1)
α	Angle between axes x and z
Γ	$V_\varphi r$
ξ	Curvilinear coordinate along meridional streamline
Φ	Dissipation function
φ_0	φ-distribution $\varphi = \varphi_0(y)$ or $\varphi = \varphi_0(\psi)$ specified along a selected initial line $x = x_0(y)$ or $\xi = \xi_0(\psi)$
ν	Angle between the integral path and the ξ-axis
ψ	Stream function
ω	Angular speed of the rotor
ρ	Density
$\bar{\partial}/\partial$	Partial derivatives along S_2 surface[1]

Subscripts

0	Reference state
pr	Prescribed
l	Meridional streamwise component
t	Tangential component
x, y	Components along x and y coordinates
$+, -$	Values on the two sides of the integration path

Y-momentum equation:

$$\frac{\partial W_y}{\partial x}-\frac{\partial W_x}{\partial y}=Br\rho\left(W_\varphi\frac{\partial W_\varphi}{\partial\psi}-\frac{\partial\mathring{R}}{\partial\psi}+\frac{mp}{\rho}\frac{\partial\tilde{S}}{\partial\psi}\right)+\tilde{F}_y \tag{2}$$

Equation of state:

$$p=C_0\rho^k\exp(\Delta\tilde{S}) \tag{3}$$

For an adiabatic flow, the energy equation is

$$kmp/\rho+(W^2-u^2)/2=\mathring{R}(\xi,\psi) \tag{4}$$

Here $\tilde{F}_y=(F_y+f_y)/W_x$ and is determined from ψ and ρ of the previous iteration. $\mathring{R}(\xi,\psi)$ is a given distribution of rothalpy, and the transformation

$$\xi=x,\qquad \psi=\psi(x,y) \tag{5}$$

is introduced. In addition, we specify the distributions of $V_\varphi r$, B, and S along S_2: $V_\varphi r=\Gamma(\xi,\psi)$, $S=S(\xi,\psi)$, and $B=B(x,y)$.

Note that henceforth, for brevity, all partial derivatives along S_2 are symbolized as conventional ones.

Boundary conditions

Along some part of the external boundary (inlet, outlet, hub and casing walls) ∂A_1, $\psi=\psi_{pr}$ is given, but along the remaining part ∂A_2, $W_t=(W_t)_{pr}$ is specified.

The composition of ∂A_1 and ∂A_2 may be quite different from case to case, depending on the design requirements under consideration. Three typical options are listed in Table 1.

All physical parameters are required to be continuous at internal interfaces C_i and artificial interfaces ∂A_3.[16]

Variational principles

By using the inverse-derivation method and the constraint-removing transformation[2,12], we can derive variational principles for the semi-inverse problem of an S_2 stream sheet in mixed flow turbomachines, as shown in Table 2. Here SGVP and GVP denote a subgeneralized VP and a generalized VP, respectively. In all VPs in Table 2 all boundary conditions have been converted to natural ones. This will greatly facilitate the practical numerical handling of the quite complicated boundary conditions under consideration.

Table 1 Typical compositions of the boundary

	Option I	Option II	Option III
∂A_1	$h+t$*	$h+t+C_e$	$h+t+C_e+C_a$
∂A_2	C_e+C_a	C_a	—

* For symbols see Nomenclature.

Family of VPs for the H_A problem

Y-momentum equation and continuity equation on an image plane

Assuming that the inversion of the transformation, Equation 5 exists, we have

$$x=\xi,\qquad y=y(\xi,\psi) \tag{6}$$

Then the original domain with partly unknown boundary is transformed into one with fully known boundary. In fact, it becomes a trapezoid domain if there is no injection and/or suction along the hub and casing walls. The Y-momentum equation and continuity equations on the image plane (ξ,ψ) can be written as follows:

$$Br\rho W_x\frac{\partial y}{\partial\psi}=1,\qquad \frac{\partial y}{\partial\xi}=Br\rho W_y\frac{\partial y}{\partial\psi} \tag{7}$$

$$\frac{\partial W_y}{\partial\xi}-Br\rho W_t\frac{\partial W_t}{\partial\psi}=Br\rho\left\{\left(\frac{\Gamma}{r^2}-\omega\right)\frac{\partial\Gamma}{\partial\psi}-\frac{\partial\mathring{R}}{\partial\psi}+\frac{mp}{\rho}\frac{\partial S}{\partial\psi}\right\}+\tilde{F}_y \tag{8}$$

Equations 3 and 4 remain valid here.

Boundary conditions

Let $\partial\bar{A}_1$ denote a part of the external boundary, on which $P=(P)_{pr}$ is given, and $\partial\bar{A}_2$ the remaining part, on which $Y=(Y)_{pr}$ is given. $W_y=(W_y)_{pr}$ must be given at the inlet and outlet boundaries.

Let $\bar{C}_i$ denote the internal interfaces, and $\partial\bar{A}_3$ the artificial interfaces, on which all physical parameters should be continuous.

Variational principles

A family of VPs for the H_A problem on the image plane, ξ-ψ, which is shown in Figure 1(b), can be derived from the foregoing

Table 2 VPs for the semi-inverse problem

	Functionals	Constraints	Euler's equations
VP_1	$J_1(\psi)=\int_A\left\{\frac{1}{2Br\rho}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]+Br\left[\frac{\rho}{2}(u^2-W_\varphi^2)-C_0m\rho^k e^{(\Delta\tilde{s})}+\rho\mathring{R}\right]-\tilde{F}_y\psi\right\}dx\,dy+S_b$	(1), (3), (4)	(2)
$SGVP_2$	$J_2(\psi,\rho)=J_1(\psi)$ formally	(1), (3)	(2), (4)
GVP_3	$J_3(\psi,\rho,p,W_x,W_y)=\int_A\left\{\left(W_x\frac{\partial\psi}{\partial y}-W_y\frac{\partial\psi}{\partial x}\right)+Br\rho\left[\frac{1}{2}(u^2-W^2)+\mathring{R}\right]-Brmp\left(1+\Delta\tilde{s}-\ln\frac{p}{C_0\rho^k}\right)-\tilde{F}_y\psi\right\}dx\,dy+S_b$	—	(1), (2), (3), (4)
	$s_b=\int_{\partial A_1}(\psi-\psi_{p_r})\mathbf{W}\cdot d\mathbf{s}+\int_{\partial A_2}(W_t)_{p_r}\psi\,ds+\int_{\partial A_3}(\psi_--\psi_+)\mathbf{W}_+\cdot d\mathbf{s}$		

Table 3 VPs for the H_A problem

	Functionals	Constraints	Euler's equations
VP_4	$J_4(y)=\int_{(\bar{A})}\left\{\frac{[1+(\partial y/\partial\xi)^2]}{2Br\rho(\partial y/\partial\psi)}+Br\rho\frac{\partial y}{\partial\psi}[\mathring{R}-(W_\varphi^2-u^2)-C_0m\rho^{k-1}\exp(\Delta\tilde{s})]+\tilde{F}_y y\right\}d\xi\, d\psi+S'_b$	(3), (4), (7)	(8)
$SGVP_5$	$J_5(y,\rho)=J_4(y)$ formally	(3), (7)	(4), (8)
GVP_6	$J_6(y,\rho,p,W_x,W_y)=\int_{(\bar{A})}\left\{W_x+W_y\frac{\partial y}{\partial\xi}+Br\rho\left[\mathring{R}+\omega\Gamma-\tfrac{1}{2}(\Gamma^2/r^2+W_f^2)-\frac{mp}{\rho}\left(1+\Delta\tilde{s}-\ln\frac{p}{C_0\rho^k}\right)\right]\frac{\partial y}{\partial\psi}+\tilde{F}_y y\right\}d\xi\, d\psi+S'_b$	—	(3), (4), (7), (8)

$$S'_b=\int_{(\partial\bar{A}_1)}(Bp)_{pr}(ry-y^2\cos\alpha/2)\cos\nu\, ds-\int_{(\bar{C}_e,\bar{C}_a)}(W_y)_{pr}y\sin\nu\, ds+\int_{(\partial\bar{A}_2)}(y-y_{pr})(Brp\cos\nu-W_y\sin\nu)\, ds$$
$$+\int_{(\partial\bar{A}_3)}(y_- - y_+)(Brp\cos\nu-W_y\sin\nu)_+\, ds,\qquad r=x\sin\alpha+y\cos\alpha$$

J_1–J_3 by a transformation of inversion.[2] This family is shown in Table 3 where $B(x,y)$, $\Gamma(\xi,\psi)$, $\mathring{R}(\xi,\psi)$, $S(\xi,\psi)$, and $\tilde{F}_y(\xi,\psi)$ are assumed known and $W_\varphi=\Gamma/r-\omega r$.

Again, in Table 3 all boundary conditions have been converted into natural ones.

Two families of nonclassical VPs for the H_A problem

Restricted VPs

Restricted VPs can be generated by adding

$$\int_{(\bar{A})}\mathring{p}\left(yr-y^2\cos\frac{\alpha}{2}\right)\frac{\partial B}{\partial\psi}\, d\psi\, d\xi$$

to each of the functionals J_4–J_6, where $\mathring{p}$ meas that this pressure $\mathring{p}$ must be held fixed when the variations of J_4–J_6 are being made and $\mathring{p}$ is set to p immediately after variation.[18] In this family of VPs, $B=B(\xi,\psi)$ is specified.

VPs with iterative terms

VPs with iterative terms can be obtained by adding

$$\int_{(\bar{A})}p^{(n-1)}\left(yr-y^2\cos\frac{\alpha}{2}\right)\frac{\partial B}{\partial\psi}\, d\psi\, d\xi$$

to each of the functionals J_4–J_6, where $p^{(n-1)}$ denotes p of the previous iteration; $B=B(\xi,\psi)$ is also a known function for this family.

Unification of VP families for axial, radial, and mixed flow turbomachines

Families of VPs for the same problems of axial and radial flow turbomachines derived in previous papers[3,4] can be regarded as special cases of the present results. Let $y=r$, $x=z$ (i.e., $\alpha=0$), and $y=z$, $x=-r$ (i.e., $\alpha=-\pi/2$), in J_1–J_6. Then the functionals of Refs. 3 and 4 can be derived.

Determination of viscous terms

To determine viscous terms, an approximate loss model has been suggested in Refs. 3 and 4, starting from the first law of thermodynamics,[2,15]

$$\frac{D\mathring{R}}{Dt}=\frac{Dq'}{Dt}+\frac{\Phi}{\rho}+\mathbf{f}\cdot\mathbf{W} \tag{9a}$$

$$\frac{Dq'}{Dt}=\frac{DU}{Dt}+p(d\rho^{-1})-\frac{\Phi}{\rho} \tag{9b}$$

and the Gibbs identity,

$$\frac{p}{R\rho}\frac{DS}{Dt}=\frac{DU}{Dt}+pd\rho^{-1} \tag{10}$$

Combining Equations 9 and 10 gives

$$\frac{p}{R\rho}\frac{DS}{Dt}=\frac{Dq'}{Dt}+\frac{\Phi}{\rho} \tag{11}$$

$$\frac{D\mathring{R}}{Dt}=\frac{Dq'}{Dt}+\frac{\Phi}{\rho}+\mathbf{f}\cdot\mathbf{W}=\frac{p}{R\rho}\frac{DS}{Dt}+\mathbf{f}\cdot\mathbf{W} \tag{12}$$

Consider adiabatic flow. We know from boundary layer theory that for adiabatic flow with $Pr=1$ the stagnation enthalpy H of steady absolute flow is uniform, including, of course, $DH/Dt=0$.[2] For adiabatic steady relative flow with $Pr=1$, however, the rothalpy $\mathring{R}$ is, strictly speaking, generally not uniform. Nevertheless, $D\mathring{R}/Dt=0$ can be assumed approximately to hold. This assumption is sufficiently accurate for engineering calculations, at least for S_2 flows not deviating significantly from axial flow. Thus, from Equations 11 and 12 we get

$$\frac{p}{R\rho}\left(\frac{DS}{Dt}\right)_{ad}=\frac{\Phi}{\rho}=-\mathbf{f}\cdot\mathbf{W} \tag{13}$$

where $(DS/Dt)_{ad}$ for adiabatic flow can be evaluated using empirical loss coefficients.

Moreover, if we assume, similar to 1D flow[2], that $\mathbf{f}$ is parallel but opposite to $\mathbf{W}$, we obtain from Equation 16

$$f=\frac{\Phi}{\rho|W|}$$

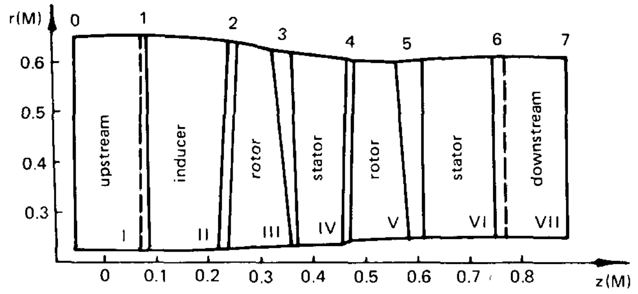


Figure 2 Meridional plane of a compressor

and

$$f_i = \frac{-fW_i}{|W|} = \frac{-\Phi}{\rho W^2} W_i \qquad (i = x, \varphi, y) \tag{14}$$

Other approximate loss models (e.g., those suggested by Bosman and Marsh[13] and Horlock[14]) can also be incorporated for determining viscous terms in the VPs.

Determination of $\tilde{F}_y$ and the S_2 stream surface shape

In all VPs presented above $\tilde{F}_y$ is assumed to be known but should be updated after every converged ψ (or y)-solution has been obtained. This updating can be accomplished in two ways. The one given in Ref. 1 does not have sufficient numerical accuracy due to multiple numerical differentiations involved, so a new method is suggested as follows:

Because $V_\varphi r = \Gamma(\xi, \psi)$ is known, the circumferential component of the momentum equation for S_2 yields[2,3,4]

$$F_\varphi + f_\varphi = \frac{1}{r}\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{W_x}{r}\frac{\bar{\partial}\Gamma}{\partial x} \tag{15}$$

If the equation of the S_2 stream sheet is $\varphi = \varphi(x, y)$, then, since **F** is orthogonal to S_2, we can write[2]

$$\frac{F_y}{F_\varphi r} = -\frac{\bar{\partial}\varphi}{\partial y}$$

which, after inserting Equation 15, gives

$$\tilde{F}_y = \frac{f_y}{W_x} + \left(\frac{f_\varphi r}{W_x} - \frac{\bar{\partial}\Gamma}{\partial x}\right)\frac{\bar{\partial}\varphi}{\partial y} \tag{16}$$

Obviously, we have

$$\frac{\bar{\partial}\varphi}{\partial l} = \frac{W_\varphi}{rW_l} = \frac{\Gamma/r^2 - \omega}{W_l}$$

Integrating it aong the streamline, we obtain

$$\varphi(\xi, \psi) = \varphi_0 + \int_{\xi_0}^{\xi} \left(\frac{\Gamma}{r^2} - \omega\right)\frac{\mathrm{d}\xi}{W_x} \tag{17}$$

We obtain the distribution of φ along the streamline. Then, $\bar{\partial}\varphi/\partial\psi$ can be calculated simply by numerical differentiation. Using

$$\frac{\bar{\partial}\varphi}{\partial y} = \frac{\bar{\partial}\varphi}{\partial\psi}\frac{\bar{\partial}\psi}{\partial y} = Br\rho W_x \frac{\bar{\partial}\varphi}{\partial\psi} \tag{18}$$

we get $\bar{\partial}\varphi/\partial y$. Consequently, $\tilde{F}_y$ can be updated from Equation 16.

In this method of updating $\tilde{F}_y$ the computation of F_φ and F_z is not required at all. Moreover, because direct use of the S_2 stream surface equation $\varphi = \varphi(x, y)$ (or $\varphi = \varphi(\xi, \psi)$) has been made, the integrability condition[1]

$$\mathbf{F} \cdot \nabla \times \mathbf{F} = 0 \tag{19}$$

need not be employed explicitly.

Computational example by FEM

Finally, a computational example of the semi-inverse problem of the S_2 stream sheet of a two-stage, axial flow air compressor, as shown in Figure 2, is given. The rotational speed is 6540 r/min. The inlet total parameters are uniform. The distribution of entropy in the flow field is determined by guessing the isentropic efficiency in the rotor region and the total pressure recovery coefficient in the remaining regions. The distribution of the stream function along the hub and tip is given and handled as essential boundary conditions. At the inlet and outlet we have the natural boundary condition $W_t = 0$. The flow field is divided into 3×24 or 5×24 eight-node isoparametric elements. The nonlinear algebraic equation system obtained from δJ_1 is linearized by the distribution of ψ and ρ of the previous iteration.

A coupled ψ-ρ iteration method is used to solve the semi-inverse problem of an S_2 stream sheet. That is, the iterations of ρ alternate with the iterations of ψ. A first guess at the distribution of ψ is made according to uniform flow in every cross section of the channel. The solution of ρ at every node point is reduced to finding the null point of a function $f(\rho)$:

$$f(\rho) = A_1\rho^{k-1} + A_2/\rho^2 + A_3 \tag{20}$$

where

$$A_1 = C_0 km \exp(\Delta\tilde{S})$$

$$A_2 = \frac{1}{2}\left(\frac{1}{Br}\right)^2\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right]$$

$$A_3 = \frac{1}{2}\left(\frac{\Gamma}{r}\right)^2 - \omega\Gamma - \mathring{R}$$

An initial guess at the distribution of ρ can be made by taking $\rho^{(0)} = -(A_3/A_1)^m$. At some node points no null points of $f(\rho)$ are found, because an unsuitable guess at the initial stage of iteration has been made. Then we developed the discriminant

$$A_2^{k-1/k+1} \leqslant (k-1)^{(k-1)/(k+1)}\left(\frac{2}{A_1}\right)^{2/(k+1)}\left(\frac{-A_3}{k+1}\right) \tag{21}$$

Null points of $f(\rho)$ cannot be found unless Equation 10 can be satisfied at that node point.

The present treatment is validated by the calculated results. The calculated axial velocity profiles at outlets of the blade domains are shown in Figure 3, and the calculated distribution of static pressure along the hub and tip casings is given in Figure 4. Figure 5 illustrates the convergence behavior of iterations for ψ. It seems that the convergence of the present treatment is satisfactory.

Further computations of examples of the H_A problem are now in progress and will be reported in a later paper. Noting, however, that the variational finite element solutions to the hybrid problems H_A and H_B of blade-to-blade flow have been given,[19] we can expect by similarity that good numerical results could also be obtained for the H_A problem of S_2 flow by FEM based on VPs derived herein.

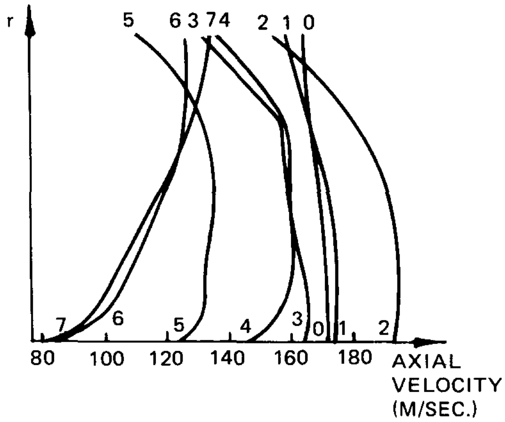


Figure 3 Axial velocity profiles at outlets of blade domains

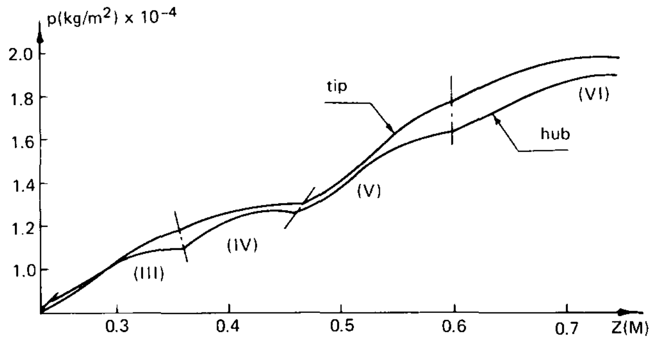


Figure 4 Distribution of static pressure along hub and tip casings

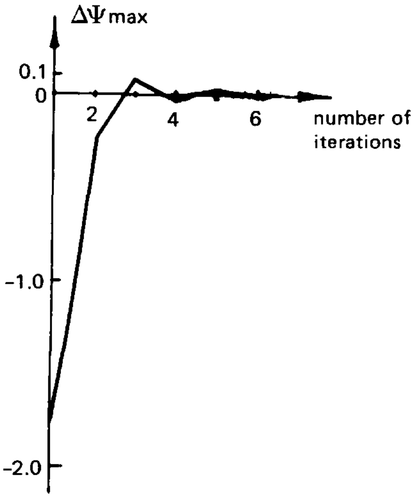


Figure 5 Convergence of iteration process

Conclusions

Families of VPs for the semi-inverse, H_A hybrid and complete inverse problems of an S_2 stream sheet in mixed flow turbomachines are developed unifying the results of previous papers.[3,4] Thus, a more perfect theoretical basis is provided for constructing a universal computer program for FEM or other direct variational methods. Certainly, its utilization will bring great convenience into practical calculations. From the computational example, it is seen that the application of FEM based on VPs derived here to the problems of an S_2 stream sheet is reliable. The theory developed here allows all or part of the inner and outer annular walls to be profiled •rationally by specifying favorable pressure distributions, providing broader and versatile ways for blading design.

Acknowledgment

This project is supported by the National Natural Science Fund of the People's Republic of China.

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